Poset persistence as filtration of weighted objects

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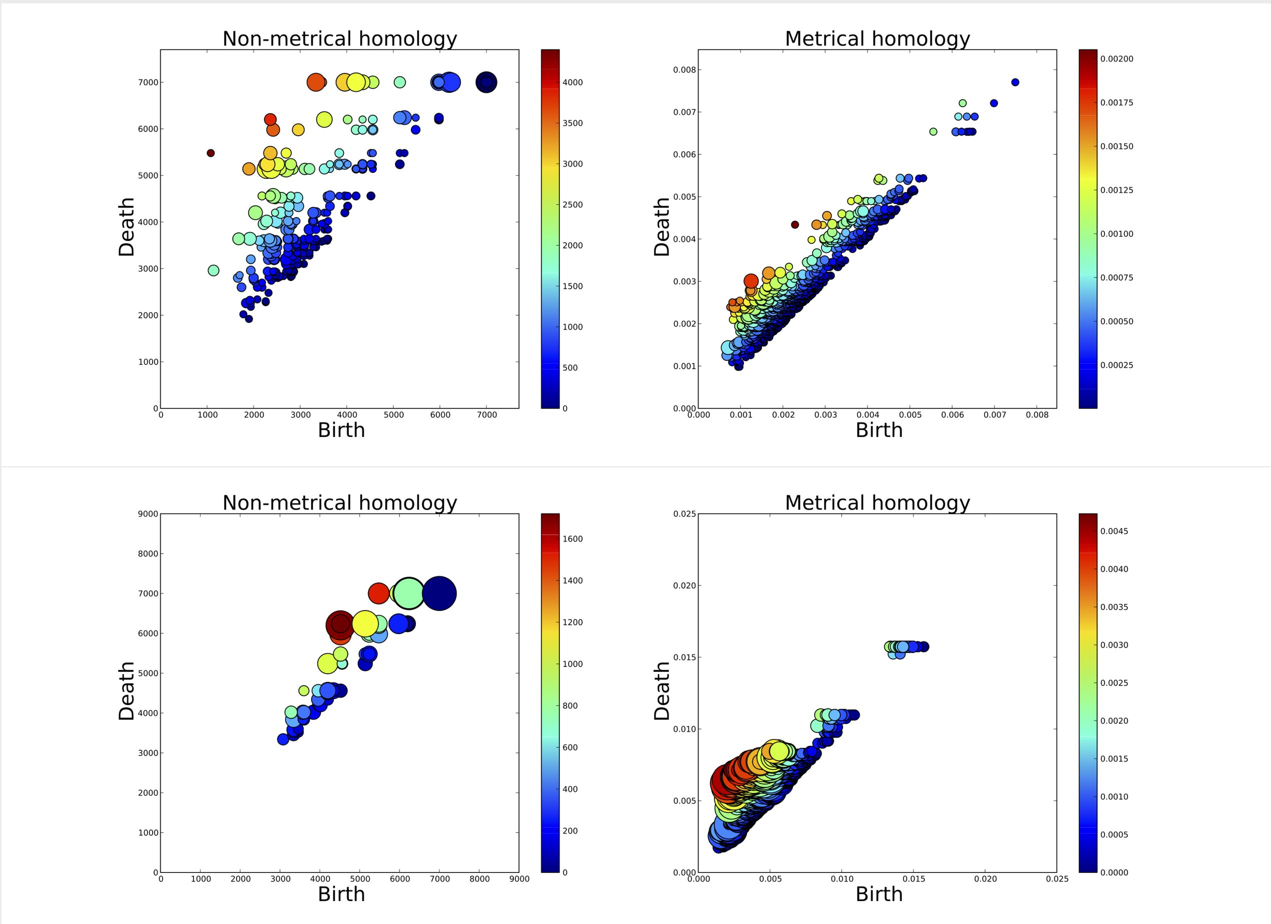
Motivation

We give a mathematical justification to the approach introduced in [1], showing that in fact for every P-persistent object it is possible to build a weighted graph whose weighted clique filtration has the same persistent homology of the original space.
The interest in this approach is substantiated by the empirical resolution limits found employing distances on networks. In fact, by comparing the results obtained using metric-based persistent homology and the weighted clique homology, it is easy to see that a metrical approach on complex networks tends to be blind to most interesting features in complex networks.



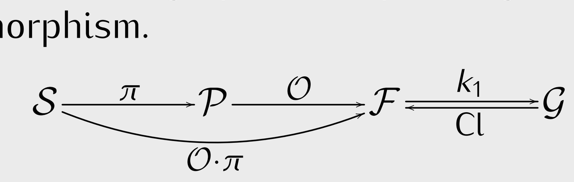
Comparison of results obtained for H_1 (*top*) and H_2 (*bottom*) on the Sociopatterns network with weighted graph filtration (*left*) and metrical filtration (*right*) as described in the main text.

Useful definitions I : Categories

- $\mathcal{T}_f$ Category of finite topological spaces with continuous maps.
- $\mathcal{T}_f^0$ Category of finite topological spaces satisfying the T_0 condition with continuous maps.
- $\mathcal{P}$ Category of finite posets $P = (P, \leq)$ with order preserving maps.
- $\mathcal{S}$ Category of simplicial complexes with simplicial maps.
- $\mathcal{F}$ Category of flag complexes with simplicial maps.
- $\mathcal{G}$ Category of reflexive graphs with edge preserving morphisms.
- Vect_k Category of k-vector spaces, k a fixed field.

Useful definitions II: Functors

- $\mathcal{T}_f \longrightarrow \mathcal{T}_f^0$ the *Kolmogorov quotient.*
- $\mathcal{T}_f^0 \longleftrightarrow \mathcal{P}$ is an isomorphism.
- $\mathcal{O} : \mathcal{P} \longrightarrow \mathcal{S}$, and $\mathcal{O}(P)$ is the *order complex* of $P \in \mathcal{P}$.
- $\pi : \mathcal{S} \longrightarrow \mathcal{P}$, and $\mathcal{O}(\pi(\Sigma))$ is the *barycentric subdivision* of $\Sigma \in \mathcal{S}$.
- $\mathrm{Cl} : \mathcal{G} \longrightarrow \mathcal{S}$ where $[v_0, \ldots, v_k] \in \mathrm{Cl}(G) \leftrightarrow (v_i, v_j) \in E \ \forall i \neq j$.
- $k_1 : \mathcal{S} \longrightarrow \mathcal{G}$ where $k_1(\Sigma) \in \mathcal{G}$ is the graph corresponding to Σ_1.
  $\mathrm{Cl}, k_1 : \mathcal{F} \longleftrightarrow \mathcal{G}$ is an isomorphism.



- $\mathrm{H}_i : \mathcal{S} \longrightarrow \mathrm{Vect}_k$ the i^{th}-simplicial homology functor.
- $\mathrm{H}_i^{\mathrm{S}} : \mathcal{T}_f \longrightarrow \mathrm{Vect}_k$ the i^{th}-singular homology functor.

References

1. Petri G, Scolamiero M, Donato I, Vaccarino F (2013) Topological Strata of Weighted Complex Networks. PLoS ONE 8(6): e66506. doi: 10.1371/journal.pone.0066506
2. Petri G, Expert P, Turkheimer F, Carhart-Harris R, Nutt D, Hellyer PJ and Vaccarino F (2014) Homological scaffolds of brain functional networks. J. R. Soc. Interface vol. 11. doi: 10.1098/rsif.2014.0873
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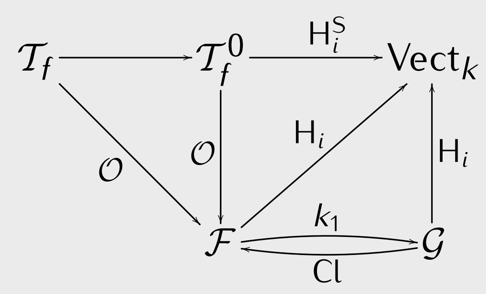
Finite Spaces Homology is Graph Homology

For all $X \in \mathcal{T}_f$:

$$\mathrm{H}_i^{\mathrm{S}}(X) \cong \mathrm{H}_i(\mathcal{O}(X)) = \mathrm{H}_i(\mathrm{Cl}(k_1(\mathcal{O}(X)))) \quad (1)$$

and



commutes. (2)

P-persistence

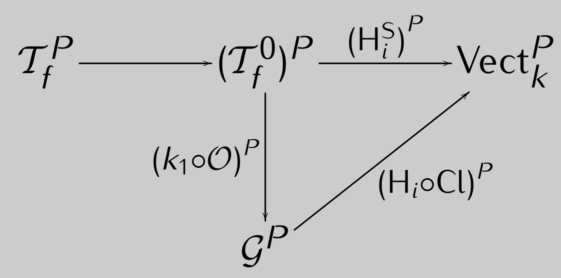
A *P-persistence object* in $\mathcal{A}$ is a functor $\varphi : P \to \mathcal{A}$, $P \in \mathcal{P}$ a poset and $\mathcal{A}$ an arbitrary category.
$\mathcal{A}^P$ Category of P-persistence objects in $\mathcal{A}$ with their natural transformation.

Proposition

Let $\tau \in \mathcal{T}_f^P$ then $\exists \theta \in \mathcal{G}^P$ such that one has the following equality of functors

$$\mathrm{H}_i^{\mathrm{S}} \circ \tau \cong \mathrm{H}_i \circ \mathrm{Cl} \circ \theta$$

Therefore P-persistence homology is P-persistence homology of graphs.



(3)

A *P-weighted graph* is a pair (G, ω), where $\omega : (G, \subseteq) \to P$ is a morphism of posets i.e. a function $G \to P$, which is continuous in the Alexandrov topology.

$\mathcal{G}_P$ Category of weighted graphs with morphisms $\alpha : (G, \omega) \to (H, \theta)$ induced by a simplicial map $\rho : G \to H$, s. t. $\alpha(G_v) \subseteq H_v$, where $\forall v \in P$, $G_v = \{x \in G \,|\, \omega(x) \leq v\}$.

Consider $(G, \omega) \in \mathcal{G}_P$, $v \in P$, we define a functor $\Phi : \mathcal{G}_P \to \mathcal{G}^P$ in the following way:

$$(G, \omega) \to \tau_G = \begin{cases} \tau_G(v) = G_v \\ \tau_{G,uv} : G_u \hookrightarrow G_v \end{cases} \qquad \forall v \in P, \forall u \in P_v \quad (4)$$

Consider $\varphi \in \mathcal{G}^P$, $v \in P$, $\varphi(v) \in \mathcal{G}$ is not a weighted graph. We define a functor $\Psi : \mathcal{G}^P \to \mathcal{G}_P$ in the following way:

$$\varphi \to (G, \omega) = \begin{cases} G = \bigsqcup_{v \in P} \varphi_v \\ \omega|_{\varphi_v} = v \end{cases} \qquad \forall v \in P \quad (5)$$

Main result

$\bar{\mathcal{G}}_P$ subcategory with $Obj(\bar{\mathcal{G}}_P) = Obj(\mathcal{G}_P)$, and morphisms the maps $\alpha : (G, \omega) \to (H, \theta)$ s.t. $\forall x \in G$, $\theta(\alpha(x)) = \omega(x)$.

Theorem $\Phi|_{\bar{\mathcal{G}}_P} \dashv \Psi$, that is $\mathrm{hom}_{\bar{\mathcal{G}}_P}((X, \omega), \Psi(\varphi)) \cong \mathrm{hom}_{\mathcal{G}^P}(\Phi((X, \omega)), \varphi)$

Let $\mathcal{G}_i^P$ be the subcategory of $\mathcal{G}^P$ whose objects are $\varphi \in \mathcal{G}^P$ such that the morphisms $\varphi(u \leq v) : \varphi(u) \to \varphi(v)$ are injective maps.

Theorem $\Psi|_{\mathcal{G}_i^P} \dashv \Phi$, that is $\mathrm{hom}_{\mathcal{G}_P}(\Psi(\varphi), (X, \omega)) \cong \mathrm{hom}_{\mathcal{G}_i^P}(\varphi, \Phi((X, \omega)))$

Conclusions

Combining the Proposition and Theorem above, we obtain that for all $\tau \in (\mathcal{T}_f^0)^P$, there is $(G, \omega) \in \mathcal{G}_P$ such that $\mathrm{H}_i^{\mathrm{S}} \cdot \tau \cong \mathrm{H}_i(G, \omega)$.